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General pseudoadditivity of composable entropy prescribed by the existence of equilibrium

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The concept of composability states that entropy of the total system composed of independent subsystems is a function of entropies of the subsystems. Here, the most general pseudoadditivity rule for composable entropy is derived based only on the existence of equilibrium.

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I. INTRODUCTION

Consider a thermodynamic system composed of two independent subsystems, A and B, in contact with each other. The concept of ''composability'' [1] states that total entropy of any kind, $S(A,B)$, is given in terms of the entropies of the subsystems, $S(A)$ and $S(B)$,

$$S(A,B)=f(S(A),S(B)), \tag{1}$$

where f is a certain bivariate function of the C^2 class and is assumed to be symmetric

$$f(S(A),S(B))=f(S(B),S(A)). \tag{2}$$

This concept may be regarded as a generalized composition rule for the total entropy. The celebrated Boltzmann-Shannon entropy possesses the additivity property [2]

$$f(S(A),S(B))=S(A)+S(B). \tag{3}$$

Another example may be supplied by the rule [3]

$$f(S_q(A),S_q(B))=S_q(A)+S_q(B)+Q(q)S_q(A)S_q(B), \tag{4}$$

where $Q(q)$ is a function of the ''entropic index'' q, satisfying $Q(1)=0$. $Q(q)=1-q$ and $Q(q)=q-1$ correspond to the Tsallis entropy [1,4] and the modified Tsallis entropy [3,5], respectively. In both cases, the deviation of q from unity measures the degree of nonextensivity of the entropies. Clearly, additivity in Eq. (3) is recovered in the limit $q\to 1$. The property in Eq. (4) is referred to here as Tsallis-type pseudoadditivity.

To construct thermodynamics based on entropy, it is necessary to define the equilibrium state, first. This is essentially relevant to the zeroth law of thermodynamics. In this paper we derive the most general form of pseudoadditivity of composable entropy based only on the existence of equilibrium. We shall see how the structure in Eq. (4), that is, the sum and the product of two entropies, is universal.

II. EQUILIBRIUM CONDITION AND GENERAL PSEUDOADDITIVITY OF ENTROPY

Let $X(A)$ and $X(B)$ be certain extensive variables of the subsystems A and B, respectively. X may be, for example, an internal energy, system volume, or particle number. Then, the equilibrium state [6] characterized by the maximum of the total entropy in Eq. (1) with the fixed total amount, $X(A,B)=X(A)+X(B)$, yields

$$\frac{\partial f(S(A),S(B))}{\partial S(A)}\frac{dS(A)}{dX(A)}=\frac{\partial f(S(A),S(B))}{\partial S(B)}\frac{dS(B)}{dX(B)}. \tag{5}$$

To establish the zeroth law of thermodynamics, by which a physical variable (such as temperature, pressure, or chemical potential) common to the subsystems in equilibrium can be identified, it is essential to be able to realize factorization of Eq. (5) in the form

$$F(A)=F(B). \tag{6}$$

This means that the following set of equations should hold:

$$\frac{\partial f(S(A),S(B))}{\partial S(A)}=k(S(A),S(B))g(S(A))h(S(B)), \tag{7}$$

$$\frac{\partial f(S(A),S(B))}{\partial S(B)}=k(S(A),S(B))h(S(A))g(S(B)). \tag{8}$$

Here g and h are some functions and k is a differentiable bivariate function. In particular, h has to be differentiable. The function k does not have the factorized form, in general. The symmetry of f shown in Eq. (2) tells us that k is also symmetric

$$k(S(A),S(B))=k(S(B),S(A)). \tag{9}$$

Since f is of the C^2 class, the integrability condition holds, leading to

$$\begin{aligned}&\frac{\partial k(S(A),S(B))}{\partial S(A)}h(S(A))g(S(B))\\&\quad+k(S(A),S(B))\frac{dh(S(A))}{dS(A)}g(S(B))\\&=\frac{\partial k(S(A),S(B))}{\partial S(B)}g(S(A))h(S(B))\\&\quad+k(S(A),S(B))g(S(A))\frac{dh(S(B))}{dS(B)}\end{aligned} \tag{10}$$

Using Eqs. (7) and (8), we rewrite this equation as follows:

$$\frac{1}{k^2(S(A),S(B))}\frac{\partial k(S(A),S(B))}{\partial S(A)}\frac{\partial f(S(A),S(B))}{\partial S(B)}+\frac{dh(S(A))}{dS(A)}g(S(B))$$
$$=\frac{1}{k^2(S(A),S(B))}\frac{\partial k(S(A),S(B))}{\partial S(B)}\frac{\partial f(S(A),S(B))}{\partial S(A)}+g(S(A))\frac{dh(S(B))}{dS(B)}. \tag{11}$$

This is an identity and therefore leads to the following equations for the factorized and nonfactorized parts:

$$\frac{dh(S(A))}{dS(A)}g(S(B))=g(S(A))\frac{dh(S(B))}{dS(B)}, \tag{12}$$

$$\frac{\partial k(S(A),S(B))}{\partial S(A)}\frac{\partial f(S(A),S(B))}{\partial S(B)}=\frac{\partial k(S(A),S(B))}{\partial S(B)}\frac{\partial f(S(A),S(B))}{\partial S(A)}, \tag{13}$$

respectively. The general solution of Eq. (13) is given by

$$k(S(A),S(B))=G(f(S(A),S(B))), \tag{14}$$

where G is an arbitrary differentiable function.

First, we consider the simplest case when k is a constant function. Without loss of generality, such a constant can be set equal to unity. Then, Eqs. (7) and (8) become

$$\frac{\partial f(S(A),S(B))}{\partial S(A)}=g(S(A))h(S(B)), \tag{15}$$

$$\frac{\partial f(S(A),S(B))}{\partial S(B)}=h(S(A))g(S(B)), \tag{16}$$

respectively. From Eq. (12), it follows that

$$\frac{1}{g(S(A))}\frac{dh(S(A))}{dS(A)}=\frac{1}{g(S(B))}\frac{dh(S(B))}{dS(B)}\equiv\lambda, \tag{17}$$

where λ is a separation constant. Using Eq. (17), we rewrite Eqs. (15) and (16) as

$$\frac{\partial f_\lambda(S(A),S(B))}{\partial S(A)}=\frac{1}{\lambda}\frac{dh_\lambda(S(A))}{dS(A)}h_\lambda(S(B)), \tag{18}$$

$$\frac{\partial f_\lambda(S(A),S(B))}{\partial S(B)}=\frac{1}{\lambda}h_\lambda(S(A))\frac{dh_\lambda(S(B))}{dS(B)}. \tag{19}$$

Here, the case $\lambda=0$ has to be interpreted as the limit $\lambda\to0$. Integrating these equations, we find

$$f_\lambda(S(A),S(B))=\frac{1}{\lambda}h_\lambda(S(A))h_\lambda(S(B))+\text{const}. \tag{20}$$

To have the convergent result in the limit $\lambda\to0$, without loss of generality, we can set the constant term in Eq. (20) equal to $-1/\lambda$ and simultaneously impose the condition

$$\lim_{\lambda\to0}h_\lambda(S)=1, \tag{21}$$

for $\forall S$. Thus, we obtain

$$f_\lambda(S(A),S(B))=\frac{h_\lambda(S(A))h_\lambda(S(B))-1}{\lambda}. \tag{22}$$

This is the form of the function of composability prescribed by the existence of equilibrium.

If both of the subsystems are at the completely ordered states, then $S(A,B)=0$. This means that

$$h_\lambda(0)=1 \tag{23}$$

for $\forall\lambda$. On the other hand, if only the subsystem B is at the completely ordered state, then we have

$$S(A,B)=S(A). \tag{24}$$

Therefore, from Eq. (22), we conclude

$$\frac{h_\lambda(S(A))-1}{\lambda}=S(A), \tag{25}$$

or equivalently,

$$h_\lambda(S)=1+\lambda S. \tag{26}$$

With this form, Eqs. (21) and (23) are clearly fulfilled. Thus, we find that Eq. (22) is equivalent to Eq. (4) with the identification, $\lambda=Q$. In other words, Tsallis-type pseudoadditivity corresponds to the simplest case when k in Eqs. (7) and (8) is a constant function.

In connection with the above result, we wish to mention that in Ref. [7] a nonextensive generalization of the Shannon-Khinchin set of axioms [8] for the Boltzmann-Shannon entropy is given. The axioms presented there are as follows.

(i) $S_q(p_1,p_2,\ldots,p_W)$ is continuous with respect to all its arguments and takes its maximum for the equiprobability distribution $p_i=1/W$ $(i=1,2,\ldots,W)$.

(ii) $S_q(A,B)=S_q(A)+S_q(B|A)+(1-q)S_q(A)S_q(B|A)$.

(iii) $S_q(p_1,p_2,\ldots,p_W,p_{W+1}=0)=S_q(p_1,p_2,\ldots,p_W)$.

Here, $p_i(i=1,2,\ldots,W)$ is the probability of finding the total system in its ith state. $S_q(B|A)$ is the conditional nonextensive entropy [7] of the subsystem B given the subsystem A. Comparing this set with the Shannon-Khinchin one, the only difference between the two is in (ii). (The Shannon-Khinchin axioms are recovered from (i)–(iii) in the limit $q\to1$.) In a particular case when A and B are independent of each other, $S_q(B|A)=S_q(B)$ holds, and accordingly (ii) becomes the Tsallis-type pseudoadditivity rule in Eq. (4) with $Q(q)=1-q$. The uniqueness theorem proved in Ref. [7] states that a quantity S_q satisfying (i)–(iii) is equal to the Tsallis entropy [1,4]

$$S_q(p_1,p_2,\ldots,p_w)=\frac{1}{1-q}\left[\sum_{i=1}^{W}(p_i)^q-1\right] \tag{27}$$

with $q>0$.

Next, let us discuss the general case in Eq. (14) and set

$$G(f)=\frac{1}{\dfrac{dH(f)}{df}}, \tag{28}$$

where H is a certain differentiable function. Then, Eq. (22) is now replaced by

$$H_\lambda(S(A,B))=\frac{h_\lambda(S(A))h_\lambda(S(B))-1}{\lambda}. \tag{29}$$

Here, we are using the same notation as in Eq. (22), which may not cause any confusions. For the completely ordered subsystems, we have

$$H_\lambda(0)=\frac{h_\lambda^2(0)-1}{\lambda}. \tag{30}$$

Also, if only the subsystem B is at the completely ordered state, then we have

$$H_\lambda(S(A))=\frac{h_\lambda(S(A))h_\lambda(0)-1}{\lambda}, \tag{31}$$

or equivalently,

$$h_\lambda(S(A))=\frac{1+\lambda H_\lambda(S(A))}{h_\lambda(0)}. \tag{32}$$

Therefore, from Eqs. (29) and (32), we find

$$H_\lambda(S(A,B)) = \frac{H_\lambda(S(A))+H_\lambda(S(B))+\lambda H_\lambda(S(A))H_\lambda(S(B))-H_\lambda(0)}{1+\lambda H_\lambda(0)}. \tag{33}$$

In a particular case when $H_\lambda(0)$ can be taken to be zero, this equation becomes

$$H_\lambda(S(A,B))=H_\lambda(S(A))+H_\lambda(S(B)) +\lambda H_\lambda(S(A))H_\lambda(S(B)). \tag{34}$$

Equation (33) is the most general pseudoadditivity rule for entropy prescribed by the existence of equilibrium. Tsallis-type pseudoadditivity is recovered when H_λ is the identity function.

As an example, let us consider the choice: $H_\lambda(f)=\sqrt{f}$ and $\lambda\to 0$. In this case, Eq. (34) gives $S(A,B)=S(A)+S(B)+2\sqrt{S(A)S(B)}$. This rule might be relevant to the black hole entropy, which is proportional to the area of a black hole [9–11]. It would be of interest to reexamine black hole thermodynamics along the lines discussed above.

III. CONCLUDING REMARKS

We have derived the most general pseudoadditivity rule for composite entropy based only on the existence of equilibrium. We have shown how Tsallis-type pseudoadditivity can be obtained as the simplest case.

In the present work, composability of entropy has been taken as a basic premise. This concept puts a stringent constraint on possible forms of entropies. For example, it is known [12] that the entropy arising from the idea of quantum groups is not composable. Recalling the discussion in Sec. II, it does not seem to be possible to define the equilibrium states in the standard manner in thermodynamics if entropy does not satisfy composability. However, it is worth noting that composability assumes divisibility of the total system into independent subsystems. Actually realizability of this independence actually puts stringent constraints on physical nature of the systems. For example, if the total system contains a long-range interaction between its microscopic components, the independence of the subsystems may hardly be realized in general. In such a situation, thorough generalization of the standard thermodynamic formalism may be required.

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